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An example of Adelic Zeta function associated to Prehomogeneous vector Spaces of Parabolic Type : Case (F_4, α_1)

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Abstract

In this work, we give an example of an Adelic Zeta function, with its functional equation and its poles, associated to Prehomogeneous vector Spaces of Parabolic Type (F_4, α_1) in the spirit of the works of A.Weil ([WE 1]) and S.Rallis and G.Schiffmann ([R-S]) in the case where the fundamental invariant is a quadratic form, using well known methods of calculus of Tamagawa numbers ([MA],[WE 1]).

Introduction

Many adelic Zeta functions have been considered for Prehomogeneous Vector Spaces (abr. PV) and many general results have been established. These works begin with A.Weil in the case of a non degenerate quadratic form [WE 1], J.G.M. Mars for the case of a cubic form [MA], J.I. Igusa in the case of finitely many orbits and absolutely admissible representations, T.Shintani and D.J.Wright for the space of binary cubic forms and by A.Yukie when the vector space and the group acting have the same dimension.

K.Ying has proved the convergence of the Zeta function in almost all cases of irreducible , reduced , regular PV. A.Yukie has studied cases where the group is a product of GL_n using smoothed version of Eisenstein series. And many other works are done in this subject.

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Here we shall give the adelic Zeta function , equation and poles for a particular and simple situation of Prehomogeneous Vector Spaces of parabolic type (ab. PV of PT).

First we recall the form that the adelic Zeta function can take for PV of PT for which the fundamental character has its values in the set of square of the field, using mean function (we have always infinitely many generic orbits in this case) and we give suffisant conditions of absolute convergence for it (prop. 1). Then we apply these results in the particular case of PV of PT having (F_4, α_1) as Dynkin diagram (whith an exception), because they are a particular case of a more general situation where it is possible to give a general description of the orbits by means of some quadratic forms ([MU 1]) and it is possible to do the calculus in a general standing.
The cases considered in this paper are listed in table 1.

I Prehomogeneous vectors spaces of parabolic type ([RU 1],[RU 2])

The situation of PV of PT that we can consider is the following :
Let $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ a finite dimensional simple graded Lie algebra over a global field $\mathbb{F}$ of 0 characteristic, H_0 is the element giving the gradation :

$$\mathfrak{g}_i = \{x \in \mathfrak{g} \mid [H_0, x] = ix\}$$

G is the centralizer of H_0 in the group $Aut_0(\mathfrak{g})$ of automorphisms of $\mathfrak{g}$ ([BO 2])
G acts on $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ by adjoint action and $(G, Ad, \mathfrak{g}_1)$ (denoted infinitesimally $(\mathfrak{g}_0, \mathfrak{g}_1)$) is a geometric PV.
Let B the Killing form of $\mathfrak{g}$, then the dual PV of $(\mathfrak{g}_0, \mathfrak{g}_1)$ is $(\mathfrak{g}_0, \mathfrak{g}_{-1})$ ([RU 1]).
We assume that

1) $\mathfrak{g}_1$ is an absolutely simple $\mathfrak{g}_0$-module
2) $\mathfrak{g}_1' = \{x \in \mathfrak{g}_1 \mid (x, 2H_0, .)$ can be completed in a sl_2-triple $\} \neq \emptyset$

(2) is equivalent to the regularity of the PV because of 1) ([RU 1]))

So $(G, Ad, \mathfrak{g}_1)$ is a PV of PT regular , absolutely irreducible, having a relative invariant of minimal degree, denoted P and we call χ the corresponding character ([RU 1]).

Let S, S_∞, S_f respectively the set of places,infinite places, finite places of the number field $\mathbb{F}$.
To every $v \in S_f$ we associate as usual $\mathfrak{O}_v = \{x \in \mathbb{F}_v \mid |x|_v \leq 1\}$, $\mathfrak{O}_v^*$ the set of unities, q_v the number of elements of the residual field. $\mathbb{A}, \mathbb{A}^*$ are respectively the ring of adeles of $\mathbb{F}$, the ideles of $\mathbb{F}$.
Let L a lattice in $\mathfrak{g}_1$, for all v in S_f , L_v is the closure of L in $\mathfrak{g}_{1,v} = \mathfrak{g}_1 \otimes_{\mathbb{F}} \mathbb{F}_v$, $L_v^0 = \{x \in L_v \mid |P_v(x)|_v = 1\}$, $K_v = \{g \in G_v \mid g(L_v) = L_v\}$ and we denote by $G_{\mathbb{A}}$ the adele group of G, K is a compact subgroup of $G_{\mathbb{A}}$ containing $\prod_{v \in S_f} K_v$,

$$\mathfrak{g}_{1,\mathbb{A}} = (\mathfrak{g}_{\mathbb{A}})_1 = \mathfrak{g}_1 \otimes_{\mathbb{F}} \mathbb{A}$$

$$\mathfrak{g}'_{1,\mathbb{A}} = \{x \in \mathfrak{g}_{1,\mathbb{A}} \mid \forall v \in S \; P_v(x_v) \neq 0 \text{ , for almost all } v \in S_f \;\; x_v \in L_v^0\}$$

$S(\mathfrak{g}_{1,\mathbb{A}})$ is the Schwartz space of functions on $\mathfrak{g}_{1,\mathbb{A}}$.
When $\chi(G) = \mathbb{F}^{*2}$ we can consider the following Zeta function :

II An adelic Zeta function under some assumptions : case $\chi(G) = \mathbb{F}^{*2}$

1- The mean function : recalls

a) The local case ([R-S],[RU 1],[IG 3])

Let $t \in \mathbb{F}_v^*$, $U_t = \{x \in \mathfrak{g}_{1,v} \mid P_v(x) = t\}$, on U_t there is a gauge form θ_t defined by $\theta_t(x) = \left(\dfrac{dx}{d(P_v(x))}\right)_{x=t}$ which determines a measure on U_t denoted $\mu_{v,t}$ and if $M_f(t) = \int_{U_t} f d\mu_{v,t}$ we have for every $f \in S(\mathfrak{g}_{1,v})$ and φ in $\mathfrak{D}(P(\mathfrak{g}'_{1,v}))$ ([R-S])

$$\int_{\mathfrak{g}_{1,v}} \varphi(P_v(x))f(x)dx = \int_{P(\mathfrak{g}'_{1,v})} M_f(t)\varphi(t)dt$$

(with the volume of L_v and the volume of $\mathfrak{O}_v$ equal to 1 if $v \in S_f$)
When v is in S_f, we denote (as usual) by M_v the mean function associated to the characteristic function of the lattice L_v.

b) The global case

We assume now that the following conditions, denoted (H) are satisfied :

HYPOTHESIS (H). —
1) Almost everywhere $P_v(\mathfrak{g}'_{1,v})$ *contains* $\mathfrak{O}_v^*$
2) There is $C > 0$, $\alpha > 1$ *such that for almost all* v *in* S_f *we have for all* t *in* $\mathfrak{O}_v^*$ $\mid M_v(t) - 1 \mid \leq C.q_v^{-\alpha}$

First, for every t in $\mathbb{F}^*$ we denote , as before, $U_t = \{x \in \mathfrak{g}_1 \mid P(x) = t\}$, then by hypothesis (2) : (1) are factors of convergence of $(d\mu_{t_v})$, with $t_v = t$ for all v in S ([WE 1]).

Secondly, we consider for $t \in \mathbb{A}^*$ as usual $U_t = \{x \in \mathfrak{g}'_{1,\mathbb{A}} \mid P(x) = t\}$, we define on U_t the measure μ_t product of the local measures μ_{v,t_v} and for $f \in S(\mathfrak{g}_{1,\mathbb{A}})$ the function $M_f(t) = \int_{U_t} f \, d\mu_t$ is a borelian function on $\mathbb{A}^*$ ([R-S]) and we have the following property

$$(*) \quad M_{f(g.)}(t) = |\chi(g)|^{-\kappa+1} M_f(\chi(g).t) \quad \text{where} \quad \kappa = \frac{\dim(\mathfrak{g}_1)}{\text{degree of P}}$$

Now we can define the adelic Zeta function : for $s \in \mathbb{C}$, $f \in S(\mathfrak{g}_{1,\mathbb{A}}), \lambda$ a unitary character of $\mathbb{A}^*$, trivial on $\mathbb{F}^*$ let

$$W_f(\lambda, s) = \sum_{\xi \in \mathbb{F}^*/_{\mathbb{F}^{*2}}} \int_{\mathbb{A}^*} M_f(t^2 \xi) \; \lambda(t) \; |t|^{2s+2} \; d^* t_{\mathbb{A}}$$

which corresponds to an integration on $(\mathbb{A}^*)^2.P(\mathfrak{g}'_1)$, with $d^*t_{\mathbb{A}} = |D|^{-\frac{1}{2}} \prod_{v \in S} d^*t_v$, D being the discriminant of $\mathbb{F}$ and $d^*t_v = \rho_v(\frac{dt_v}{|t_v|})$, $\rho_v = (1 - \frac{1}{q_v})^{-1}$ if $v \in S_f$ and else $\rho_v = 1$.
This is the adelic Zeta function introduced by A.Weil ([WE 1]), then by S.Rallis and G.Schiffmann ([R-S]) in the case where P is a quadratic form.

2 A result about convergence

NOTATIONS. —
1) If $\mathbb{F}_v$ *is not of residual characteristic 2* , *let* $\{1, \epsilon_v, \pi_v, \pi_v.\epsilon_v\}$ *a set of representatives of* $\mathbb{F}_v^*/_{\mathbb{F}_v^{*2}}$, *with* ϵ_v *in* $\mathfrak{O}_v^*$ *and* $|\pi_v| = \frac{1}{q_v}$
2) For u *and* v *in* $\mathbb{F}_v^*/_{\mathbb{F}_v^{*2}}$ *we denote by* (u, v) *the value of the Hilbert symbol and by* χ_u *the corresponding character on* $\mathbb{F}_v^*$ $(\chi_u(v) = (u, v))$
3) Let $Z_v(\lambda, s) = \int_{L_v} \lambda(P_v(x))|P_v(x)|^s dx$ *(* L_v *having volume 1)*

4) We choose an additive character τ of $\mathbb{A}$ such that $\tau(xy)$ put in duality $\mathbb{A}$ with itself in such a manner that the discrete subgroup $\mathbb{F}$ corresponds to itself with the duality $\tau(xy)$ and for $f \in S(\mathfrak{g}_{1,\mathbb{A}})$, $y \in g_{-1,\mathbb{A}}$, let
$\hat{f}(y) = \int_{\mathfrak{g}_{1,\mathbb{A}}} f(x)\tau(B(xy))\, dx_{\mathbb{A}}$.

5) $\rho_{\mathbb{F}}$ is the residue in 1 of the Zeta function of $\mathbb{F}$ and $c_{\mathbb{F}} = 2^{-r}|D|^{\frac{n-1}{2}}$.

PROPOSITION 1. — *If we assume that the conditions (H1) are verified with*

i) $\forall v \in S_f$ $P_v(\mathfrak{g}'_{1,v})$ contains $\mathfrak{O}_v^*$

ii) almost everywhere $\chi_v(K_v)$ contains $\mathfrak{O}_v^{*2}$

iii) $\exists c > 0$ and $\exists \alpha > 1$ such that almost everywhere $| \int_{L_v^0}(\pi_v, P_v(x))dx | \leq c.q_v^{-\alpha}$

iv) $\exists d > 0$ and $\exists \beta > 2$ such that for almost all v in S_f and s complex number with strictly positive real part $| Z_v(\chi_{\pi_v}, s) - Z_v(\chi_{\pi_v.\epsilon_v}, s) | \leq d.q_v^{-\beta}$.

Then for every character of $\mathbb{A}^*$, trivial on $\mathbb{F}^*$, almost everywhere non ramified we have for all $f \in S(\mathfrak{g}_{1,\mathbb{A}})$
1) for every complex number s, with strictly positive real part $W_f(\lambda, s)$ is absolutely convergent and for all $A > 0$, let $S_A = \{s \in \mathbb{C} \mid Re(s) > 0\ ,\ |Im(s)| < A.Re(s)\}$ then $lim_{s\to 0, s\in S_A} sW_f(\lambda, s) = 0$ if $\lambda \neq id$ and $lim_{s\to 0, s\in S_A} sW_f(id, s) = c_{\mathbb{F}}.\rho_{\mathbb{F}}\hat{f}(0)$
2) $\forall t \in \mathbb{A}^*$ $\sum_{\xi\in\mathbb{F}^*} M_f(t^2\xi)$ is absolutely convergent.

We can remark that conditions $(H) + ii)$ are equivalent to i),ii),iii).

Proof

1) We proceed as in the work of A.Weil ([WE 1]),§ 4.5) and S.Rallis and G.Schiffmann ([R-S], comparing $W_f(\lambda, s)$ to an appropriate sum of Ono-Integrals : $Z(f; \lambda', s) = \int_{\mathfrak{g}'_{1,\mathbb{A}}} f(x)\ \lambda'(P(x)\ d'x_{\mathbb{A}}$ with $d'x_{\mathbb{A}} = |D|^{-\frac{n}{2}} \prod_{v\in S} \rho_v dx_v$.
We can deduce from a calculus analogous to that of ([WE 1]) and the results in ([WE 3],corollary 2 p.124 and corollary p.288) that $lim_{s\to 0, Re(s)>0} s.Z(f; id, s) = \rho_{\mathbb{F}}\hat{f}(0)$ and if λ is as in the proposition $lim_{s\to 0, Re(s)>0} Z(f; \lambda, s)$ exists and is non-nil.
2) We deduce 2) from 1), using Fubini-theorem and asumption ii). □

So finally we can write for $Re(s) > 0$:

$$W_f(\lambda, s) = \int_{\mathbb{A}^*/\mathbb{F}^*} \Big(\sum_{\xi\in\mathbb{F}^*} M_f(t^2\xi) \Big)\ \lambda(t)|t|^{2s+2} d^*t_{\mathbb{A}}$$

III The case $(\mathbf{F_4}, \alpha_1)$

1 The situation

We suppose from now that we are in the case (F_4, α_1) which means the following additional properties at section I.

If $\mathfrak{a}$ is any maximal toral subalgebra of $\mathfrak{g}_0$, Δ the associated root system is also graded : $\Delta_i = \{\lambda \in \Delta \mid \lambda(H_0) = i\}$, and the positive system Δ^+ is chosen so that we have $\cup_{i\geq 1}\Delta_i \subset \Delta^+$. Then, as we have $\mathfrak{g}_1' = \{x \in \mathfrak{g}_1 \mid (x, 2H_0, .)$ can be completed in a sl_2-triple $\} \neq \emptyset$, we can assume that $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ generate $\mathfrak{g}$ so the irreducible condition of I is expressed by saying that Δ is irreducible and there is only one simple root not in Δ_0, it is in Δ_1 and $\oplus_{i\geq 0}\mathfrak{g}_i$ is a maximal parabolic subalgebra of $\mathfrak{g}$ ([RU 1]).

When $\mathfrak{g}_2$ is of dimension not greater than one, there is a maximal set of orthogonal roots of Δ_1, denoted $(\lambda_i)_{1\leq i\leq n}$, such that $\sum_{1\leq i\leq n} h_i = 2H_0$, where h_i is the co-root of λ_i. The non-nil restrictions of Δ to the subalgebra $\oplus_{1\leq i\leq n}\mathbb{F}h_i$ is a root system denoted R of rank n, having the same properties than Δ : irreducibility, gradation and is also associated to a maximal parabolic subalgebra (prop. 2.6.1 and coroll. 3.1.7 of [MU 1]).
The "commutative case" corresponds to $\mathfrak{g}_2 = \{0\}$ ($\mathfrak{g}_1$ is a commutative Lie algebra) . This case is intensively studied and it is easy to study in a general standing the ordinary Zeta function, equation and poles (except in two cases) but all these cases (except two) are known by case by case examination . This is why I talk about the case (F_4, α_1) hoping that it is of some interest.

If $\mathfrak{g}_2$ is one dimensional it is easy to prove that $P(x) = B((ad(x))^4(\omega_-), \omega_-)$, where ω_- is a generator of $\mathfrak{g}_{-2}$, P is of degree four and $\chi(G) = \mathbb{F}^{*2}$ (lemme 4.1 of [MU 1]) . If we assume that the roots $(\lambda_i)_{1\leq i\leq 4}$ are strong orthogonal and of same length, then the different types (R, α_0) , α_0 being the only simple root in R_1, are given by (prop. 6.6 of [MU 2]) :

$$(B_4, \alpha_2)\ ,\ (D_4, \alpha_2)\ ,\ (F_4, \alpha_1)$$

in the Bourbaki notation ([BO 1]) (α_1 is the long simple root at the end of the diagram)

So we assume from now that the Dynkin diagram of R is of type $\mathbf{F_4}$

If $(X_{\pm\lambda_i})_{1\leq i\leq 4}$ are chosen in the corresponding root space (as in [MU 3]) then

$$(**) \quad \mathfrak{g}_1 = \oplus_{1\leq i\leq 4}(\mathbb{F}X_{\lambda_i} \oplus \mathbb{F}\omega_i) \oplus_{1\leq i<j\leq 4} E_{i,j}$$

$\omega_i = [\omega_+, X_{-\lambda_i}]$, ω_+ being a generator of $\mathfrak{g}_2$, $E_{i,j} = \{x \in \mathfrak{g}\ [h_i, x] = [h_j, x] = x\ ,\ [h_k, x] = 0\ ,\ 1 \leq k \neq i, j \leq 4\}$, d is the common dimension of each subspace $E_{i,j}$, we have $\dim(\mathfrak{g}_1) = 8 + 6d$, $\kappa = 2 + 3\frac{d}{2}$ and d can take the values $1, 2, 4, 8$ (the case $d = 8$ is treated in [IG 2]) (cf.Table 1).
P is normalized such that $P(\sum_{1\leq i\leq 4} X_{\lambda_i}) = 1$.
Let I a non-empty subset of $\{1, 2, 3, 4\}$ and $H_I = \sum_{\{i\in I\}} h_i$ then the centralizer of $\oplus_{i\notin I}\mathbb{F}h_i$ in $\mathfrak{g}$ is reductive , its semi-simple part , denoted $\mathfrak{U} = \mathfrak{U}(\oplus_{i\notin I}\mathbb{F}h_i)$ is graded by $ad(H_I)$ and $\mathfrak{U}_i = \mathfrak{U} \cap \mathfrak{g}_i$, $(\mathfrak{U}_0, \mathfrak{U}_1)$ is a PV of PT which is absolutely irreducible and commutative if $H_I \neq 2H_0$ because $\mathfrak{U}_i = \{0\}$ for $|i| \geq 2$ and $P_{H_I}(x) = P(x + \sum_{\{i\notin I\}} X_{\lambda_i})$ is then a fundamental invariant of it and its degree is $|I|$, the number of elements of I.

2 Preliminary results

Lemma 2- local results. — $R = F_4$
1) For almost every $v \in S_f$ $Z_v(\chi_{\pi_v\epsilon_v}, s) = Z_v(\chi_{\pi_v}, s)$
2) $\exists K > 0$ *such that for almost every* $v \in S_f$ *and for all* $y \in \mathfrak{O}_v^*$ *we have* $|M_v(y) - 1| \leq K.q_v^{-\frac{3}{2}}$.

Sketch of the proof

The detailed proof will appear somewhere later. We use similar methods as in [IG 2] .
1) A change of variable
Let $H = h_1 + h_2 + h_3$, $E_i(H) = \{x \in \mathfrak{g} \mid [H, x] = ix\}$.
Using (**) every x in $\mathfrak{g}_1$ can be decomposed in the form

$$x = \sum_{0\leq i\leq 3} x_i \quad \text{with} \quad x_i \in E_i(H) \cap \mathfrak{g}_1$$

so $x_0 = tX_{\lambda_4}, x_3 = u\omega_4$, $(t, u) \in \mathbb{F} \times \mathbb{F}$
If $t \neq 0$ we have

$$x = exp(ad(C))(x') \quad \text{with} \quad x' = tX_{\lambda_4} + x_2' + x_3'\ ,\ C = -\frac{1}{t}[x_1, X_{-\lambda_4}]$$

$$x_2' = x_2 - \frac{1}{2}ad(C)^2(tX_{\lambda_4}) \quad , \quad x_3' = x_3 - [C, x_2] + \frac{1}{3}(ad(C))^3(tX_{\lambda_4})$$

So if f is a $K-$invariant function and if $L'_v = \{x \in L_v \mid t \in \mathfrak{O}_v^*\}$, we have after a suitable change of variable and with the choice of Haar measures :

$$\int_{L'_v} f(x)dx = \iiint_{\{t\in\mathfrak{O}_v^*\}\times\{x\in L_v\cap E_2(H)\}\times\{u\in\mathfrak{O}_v\}} f(tX_{\lambda_4} + x + u\omega_4)dtdxdu$$

2) Two calculus

We say that v is "good" if v is finite , not of residual characteristic 2, τ_v is of order 0 and for every non empty subset I of $\{1,2,3,4\}$, P_{H_I} and all its partial derivatives have their coefficients in $\mathfrak{O}_v$ (definition given in 1,III). Let I as before, if $|P_{H_I}(x_0)|_v = 1$ for some x_0 in $L_v \cap \mathfrak{U}(\oplus_{i\notin I}\mathbb{F}h_i)_1$, denoted simply L_{v,H_I}, then using Taylor formula we have for "good" v

$$|P_{H_I}(x_0 + \pi_v.L_{v,H_I}) - P_{H_I}(x_0)|_v \leq \frac{1}{q_v} \quad \text{so} \quad |P_{H_I}(x_0 + \pi_v L_{v,H_I})|_v = 1$$

$$\text{and} \quad I(H_I, u) = \iint_{\mathfrak{O}_v\times L_{v,H_I}} \tau_v(uwP_{H_I}(x_0 + \pi_v y))dwdy$$
$$= \int_{\mathfrak{O}_v} \tau_v(uw)dw = 1 \quad \text{if} \quad u \in \mathfrak{O}_v \quad \text{else} \quad 0$$

$$I'(H_I, u) = \iint_{\mathfrak{O}_v\times L_{v,H_I}} \tau_v(u(1+\pi_v w)P_{H_I}(x_0 + \pi_v y))dwdy$$
$$= [\int_{L_{v,H_I}} \tau_v(uP_{H_I}(x_0 + \pi_v y))dy\ ] \ . \ [\int_{\mathfrak{O}_v} \tau_v(u\pi_v w)dw\ ]$$
$$= \tau_v(uP_{H_I}(x_0)) \quad \text{if} \quad |u|_v \leq q_v \quad \text{else} \quad 0$$

Every element of the form $1 + \pi_v\mathfrak{O}_v$ is a square in $\mathfrak{O}_v^*$ so it can be written on the form $\chi_{H_I}(g)$ for some g normalizing L_{v,H_I} (χ_{H_I} being the character associated to P_{H_I}) so

$$J(H_I, u) = \iint_{\mathfrak{O}_v\times L_{v,H_I}} \tau_v(u(1+\pi_v t)P_{H_I}(x))dtdx$$
$$= \int_{L_{v,H_I}} \tau_v(uP_{H_I}(x))dx = M^*_{P_{H_I}}(u)$$

3) Proof of 1) : we assume that v is "good".
Let $M_v^*(u) = \int_{L_v} \tau_v(u.P_v(x))dx_v$, as $\mathfrak{O}_v^{*2} \subset \chi(K_v)$ and with relation (*) we see that M_v^* is $\mathfrak{O}_v^{*2}$ invariant; it is not difficult to prove that M_v^* is in $L^1(\mathbb{F}_v)$ so we have $\widehat{M_v} = M_v^*$ and for $\omega/_{\mathfrak{O}_v^*} = \chi_{\pi_v}$ we have

$$Z_v(\omega) = \int_{\mathbb{F}_v} M_v(t)\omega(t)dt = q_v^{\frac{1}{2}} C(\chi_{\pi_v}) \sum_0^{+\infty} \omega(\pi_v)^k b_{k+1}$$

$C(\chi_{\pi_v})$ being a Gauss sum ([S-T]) and $b_k = \int_{\mathfrak{O}_v^*} M^*(\pi_v^{-k}.u)\chi_{\pi_v}(u)du$, so 1) of the lemma is equivalent to show that $b_{2k} = 0$ for all $k \geq 1$.
First , for every $x \neq 0$ in L_v , there are $u_i \in \mathfrak{O}_v^* \cup \{0\}$, $z \in \pi_v.L_v$ and $k \in K_v$ such that $kx = \sum_{i=1}^4 u_i X_{\lambda_i} + z$, with $u_4 \neq 0$, so we can write

$$M_v^*(u) = \sum_{\overline{x} \in L_v/\pi_v L_v} \int_{\pi_v L_v} \tau_v(uP(\overline{x} + y))dy$$

$$= q_v^{-(8+6d)} \sum_{\overline{x} \in L_v/\pi_v L_v} \int_{L_v} \tau_v(uP(\overline{x} + \pi_v y))dy = q_v^{-(8+6d)} \sum_{\overline{x} \in L_v/\pi_v L_v} I_{\overline{x}}(u)$$

with $I_0(u) = M_v^*(\pi_v^4 u)$ and we obtain 1) by induction on k if we prove that

$$\int_{\mathfrak{O}_v^*} I_{\overline{x}}(\pi_v^{-2k}u)\chi_{\pi_v}(u)du = 0 \quad \text{for} \quad \overline{x} \neq 0$$

If $\overline{x} \neq 0$ after a change of variable we have $\overline{x} = \sum_{i=1}^4 u_i X_{\lambda_i}$, let $\overline{x}' = \sum_{i=1}^3 u_i X_{\lambda_i}$, as we have assumed that $u_4 \neq 0$ we have , applying 1) :

$$I_{\overline{x}}(u) = \iiint_{\mathfrak{O}_v \times L_v \cap E_2(H) \times \mathfrak{O}_v} \tau_v(uP((\pi_v t + u_4)X_{\lambda_4} + (\pi_v x + \overline{x}') + \pi_v z\omega_4))dtdxdz$$

but we have for every x in $E_2(H) \cap \mathfrak{g}_1$:

$$P(x + sX_{\lambda_4} + z\omega_4) = sP_H(x) - \frac{1}{4}(zs)^2$$

with the choice of the basis of $\mathfrak{g}_1$. We have after a change of variable :

$$I_{\overline{x}}(u) = J_{\overline{x}}(u). \int_{\mathfrak{O}_v} \tau_v(-u\pi_v^2 z^2)dz$$

with $J_{\overline{x}}(u) = \iint_{\{t\in\mathfrak{O}_v\}\times\{x\in L_v\cap E_2(H)\}} \tau_v(u(\pi_v t + u_4)P_H(\pi_v x + \overline{x}'))dtdx$

It remains to calculate each piece :

$$\int_{\mathfrak{O}_v} \tau_v(-u\pi_v^2 z^2)dz = \alpha(-u)q_v|u|_v^{-\frac{1}{2}} \quad \text{if} \quad |u|_v > q_v^2 \quad \text{else} \quad 1 \text{ (lemma 4.3 of [R-S])}$$

and this quantity is constant on each set $|u|_v = q_v^{2k}$ ([IG 2])
$J_{\overline{x}}(u)$ depends of the rank of $\overline{x}$ =number of $u_i \neq 0$

If $rank(\overline{x}) = 4$ then $J_{\overline{x}}(u) = I'(H, uu_4) = \tau_v(uP(\overline{x}))$ if $|u|_v \leq q_v$ and 0 else.

If $rank(\overline{x}) = 1$ then

$$\begin{aligned} J_{\overline{x}}(u) &= \iint_{\{t\in\mathfrak{O}_v\}\times\{x\in L_v\cap E_2(H)\}} \tau_v(u(\pi_v t + u_4)P_H(\pi_v x))dtdx \\ &= \iint_{\{t\in\mathfrak{O}_v\}\times\{x\in L_v\cap E_2(H)\}} \tau_v(\pi_v^3 u(\pi_v t + u_4)P_H(x))dtdx \\ &= \int_{\mathfrak{O}_v} M^*_{P_H}(\pi_v^3 uu_4) \text{ using } J(H, uu_4) \end{aligned}$$

as P_H is of odd degree(here 3) , $M^*_{P_H}$ is $\mathfrak{O}_v^*$-invariant so $J_{\overline{x}}(u) = M^*_{P_H}(\pi_v^3 u)$ and is constant on each subset $|u|_v = q_v^k$.

If $rank(\overline{x}) = 2$ or 3 we can assume that $\overline{x}' = \sum_{1\leq j\leq i-1} u_j X_{\lambda_i}, u_j \in \mathfrak{O}_v^*$, we decompose x relatively to $ad(H')$, with $H' = \sum_{1\leq j\leq i-1} h_j$:
$x = x_0 + x_1 + x_2$ with $x_j \in E_j(H') \cap E_2(H) \cap \mathfrak{g}_1$ (note that $E_{-1}(H') \cap \mathfrak{g}_1 = \{0\}$) and as $P_{H'}(\overline{x}') \in \mathfrak{O}_v^*$ we have using Taylor formula :

$$|P_{H'}(\overline{x}') - P_{H'}(\overline{x}' + \pi_v x_2)|_v \leq \frac{1}{q_v} \quad \text{and} \quad |P_{H'}(\overline{x}' + \pi_v x_2)|_v = 1$$

so we can use a change of variable like in 1), let $H" = H - H'$

$$\begin{aligned} J_{\overline{x}}(u) &= \iiint_{F_{2,0}\times F_{0,2}\times\mathfrak{O}_v} \tau_v(u(\pi_v t + u_4)P_{H'}(\overline{x}' + \pi_v x_2)P_{H"}(\pi_v x_0))dx_2 dx_0 dt \\ &= \iiint_{F_{2,0}\times F_{0,2}\times\mathfrak{O}_v} \tau_v(\pi_v^{4-i} u(\pi_v t + u_4)P_{H'}(\overline{x}' + \pi_v x_2)P_{H"}(x_0))dx_2 dx_0 dt \\ &= \iint_{F_{2,0}\times F_{0,2}} \tau_v(\pi_v^{4-i} uu_4 P_{H'}(\overline{x}' + \pi_v x_2)P_{H"}(x_0))dx_2 dx_0 \end{aligned}$$

(using $J(H", uu_4)$)

with $F_{2,0} = L_v \cap \mathfrak{U}(\oplus_{l=i}^4 \mathbb{F}h_l)_1)$ and $F_{0,2} = L_v \cap \mathfrak{U}(\oplus_{l\in\{1,\dots,i-1,4\}}\mathbb{F}h_l)_1)$
If $i = 2$ (resp. $i = 3$) $F_{2,0} = \mathfrak{O}X_{\lambda_1}$ (resp. $F_{0,2} = \mathfrak{O}X_{\lambda_3}$) and $P_{H"}$ (resp. $P_{H'}$) is a quadratic form , so

For $i = 2$

$$\begin{aligned} J_{\overline{x}}(u) &= \iint_{\mathfrak{O}_v \times F_{0,2}} \tau_v(u\pi_v^2 u_4(\pi_v s + u_1)P_{H"}(x_0))ds dx_0 \\ &= \int_{F_{0,2}} \tau_v(u\pi_v^2 u_4 u_1 P_{H"}(x_0))dx_0 \text{ (using } J(H", \pi_v^2 u u_1 u_4\ )\ ) \\ &= \gamma(uu_1u_4P_{H"})q_v^{d+2}|u|^{-(\frac{d}{2}+1)} \quad \text{if} \quad |u|_v > q_v^2 \quad \text{else} \quad 1 \end{aligned}$$

with prop.4.4. of [R-S], which gives that $J_{\overline{x}}$ is constant on the subset $|u|_v = q_v^{2k}$ ([IG 2] and prop.1.7 of [R-S])

If $i = 3$ we have as precendetly $|P_{H'}(\overline{x}' + \pi_v x_2)|_v = 1$ so

$$\begin{aligned} J_{\overline{x}}(u) &= \iint_{\mathfrak{O}_v \times F_{2,0}} \tau_v(uu_4\pi_v s P_{H'}(\overline{x}' + \pi_v x_2))ds dx_2 \\ &= I(H', uu_4\pi_v) \\ &= 1 \quad \text{if} \quad |u|_v \leq q_v \quad \text{else} \quad 0 \end{aligned}$$

Remark If we put together the results ,we obtain a formula for $M_v^*(u) - q_v^{-(8+6d)} M_v^*(\pi_v^4 u)$ which is analogous to that of § 4 of [IG 2].

4) For the proof of 2), as we have for y in $\mathfrak{O}_v^*$: $M_v(y) - 1 =$

$$-\frac{1}{2}(M_v^*(\pi_v^{-1}) + M_v^*(\pi_v^{-1}\epsilon_v)) + \frac{1}{2}\chi_{\pi_v}(y)C(\chi_{\pi_v})q_v^{\frac{1}{2}}(\ M_v^*(\pi_v^{-1}) - M_v^*(\pi_v^{-1}\epsilon_v))$$

it is enough to prove that for y in $\mathfrak{O}_v^*$ we have $|M_v^*(\pi_v^{-1}y)|_v \leq K q_v^{-2}$.
We do this with the usual methods, splitting the integral in order to make appear irreducible polynomials and then we use the lemma 1 of [L-W], for this we use changes of variable like in 1) and we complete with the value of $\int_{\mathfrak{O}_v} \tau_v(uz^2)dz$. □

This lemma proves that the hypothesis of proposition 1 are verified; we can consider $W_f(\lambda, s)$ for λ, s, f as in proposition 1 .
Let H be the Kernel of χ. If U is a subgroup of $Aut(\mathfrak{g})$ and η is an element of $\mathfrak{g}$, U_η is the centralizer of η in U.

PROPOSITION 3. — *Orbital results*
1) The singular G and H-orbits are the same.
2) Let $\xi \in \mathbb{F}^*, U_\xi = \{x \in \mathfrak{g}_1 \mid P(x) = \xi\}$ then H has a finite number of orbits in U_ξ and if Ω_ξ is a set of representatives , we have $(U_\xi)_{\mathbb{A}} = \{x \in (\mathfrak{g}_1)_{\mathbb{A}} \mid P(x) = \xi\} = \cup_{\eta\in\Omega_\xi} H_{\mathbb{A}}.\eta$, this union being disjoint, and every $H_{\mathbb{A}}.\eta$ is open in $(U_\xi)_{\mathbb{A}}$.
3) For every non singular η, the centralizer H_η is semi-simple
4) For any η the centralizer H_η is unimodular

Proof
1) It is the theorem 4.3.2 of [MU 3].
2) It comes from the "Hasse principle" : two elements are in the same G-orbit if and only if they are in the same G_v-orbit for all $v \in S$ (theorem 4.4.2 of [MU 3]).
3) It is a calculus because we can assume that $x = \sum_{1\leq i\leq 4} a_i X_{\lambda_i}, \prod_{1\leq i\leq 4} a_i \neq 0$ (lemma 2.3.2 of [MU 3]) and we use the explicit description of $(\mathfrak{g}_0)_x$ given in prop.3.1.3 of [MU 3].
4) For generic η, it comes from 3) and for singular elements it is prop.3.3 of [MU 4]. □

Lemma 2 and proposition 3 imply that the mean function verify for every $\xi \in \mathbb{F}^*$:

$$M_f(t^2\xi) = |t|^{2(\kappa-1)}.\ (\sum_{\eta\in\Omega_\xi} \frac{1}{\tau'(H_\eta)} \int_{H_{\mathbb{A}}/H_\eta} f(g'g\eta)dg_{\mathbb{A}}\ ) \text{ with } \chi(g') = t^2$$

and $\tau'(H_\eta) = c_\eta \tau(H_\eta)$, $\tau(H_\eta)$ being the Tamagawa number of H_η and c_η the proportionality coefficient between the two $H_{\mathbb{A}}$ -invariant measure of $H_{\mathbb{A}}/_{(H_{\mathbb{A}})_\eta}$ and μ_ξ restricted to the set $H_{\mathbb{A}}.\eta$.

For every x in $\mathfrak{g}_1$ there is g in G and $a_i \in \mathbb{F}, i = 1,...,4$ such that $gx = \sum_{1\leq i\leq 4} a_i X_{\lambda_i}$ (lemma 2.3.2 of [MU 3]), we note $rk(x)$ the number of $a_i \neq 0$.
For a real t , $[t]$ is its integer part

PROPOSITION 4. — *1) Convergence*
When $\Delta \neq R$ we assume that $d \neq 2$ then for every Schwartz function f on $\mathfrak{g}_{1,\mathbb{A}}$ the integral $\int_{H_{\mathbb{A}}/H} (\sum_{\xi\in\mathfrak{g}_1} f(g\xi)\ )dg_{\mathbb{A}}$ is absolutely convergent.

2) Measures on singular sets
Let η in $\mathfrak{g}_1$ of rank $j \in \{1,2,3\}$ then for every Schwartz function f on $\mathfrak{g}_{1,\mathbb{A}}$

$$T_\eta(f) = \int_{H_{\mathbb{A}}/(H_{\mathbb{A}})_\eta} f(g\eta)\, dg$$

is absolutely convergent and $T_\eta(f(g.)) = |\chi(g)|^{-j\frac{d}{2}-\frac{1}{2}[\frac{j+1}{2}]}T_\eta(f)$ for $g \in G_{\mathbb{A}}$.

Proof

1) There are two proofs .
If $\Delta \neq R$, $\mathfrak{g}$ is an absolutely simple split algebra and we have two cases : $d = 4$ (§ 7 type D_6 of [IG 1] with representation ρ_6) or $d = 8$ (§ 9 type E_7 of [IG 1] with representation ρ_1) ; we use the results of [IG 1] .
If $\Delta = R$, we use the lemme 5 of [WE 2]. For $i = 1, ..., 4$ and $t \in \mathbb{F}^*$, let $h_{\lambda_i}(t)$ the element of the Cartan subgroup of G associated to each root λ_i ([BO 2], chap VIII, § 1, $n°5$) and

$$A = \{h(t) = \prod_{1\leq i\leq 4} h_{\lambda_i}(t_i) \mid t = (t_1,t_2,t_3) \in (\mathbb{F}^*)^3, t_4 = (\prod_{1\leq i\leq 3} t_i)^{-1} \}$$

$w_1(t) = t_1^{-1}t_2$, $w_2(t) = t_2^{-1}t_3$, $w_3(t) = (t_2t_3)^{-2}$ are the simple roots associated to H (with Cartan subgroup A); let $A_C = \{t = (t_1,t_2,t_3) \in (\mathbb{R}^{*+})^3 \mid \omega_i(t) \leq 1$ for $i = 1,2,3\}$ then 1) is verified if we show that

$$\int_{A_C} \prod_{1\leq i<j\leq 4} sup\Big(1,(t_it_j)^{-rd}\Big). \prod_{1\leq i\leq 4} sup\Big(1,t_i^{-2r}\Big). \prod_{1\leq i\leq 4} sup\Big(1,(t_1...t_i^{-1}...t_4)^{-r}\Big).$$

$$t_1^{-r(6d+4)}.t_2^{-r(4d+4)}.t_3^{-2r(d+2)}. \prod_{1\leq i\leq 3} \frac{dt_i}{t_i} \quad \text{with} \quad r = \text{dim of } \mathbb{F} \text{ over } \mathbb{Q}$$

is absolutely convergent which is an easy calculus.
2) 1) implies 2) because each H_η is unimodular.
For $d \neq 1$ we can do it using the theory of quasi-invariant measures of homogeneous spaces as in [WE 2]. Indeed, if η is singular , we complete it in a sl_2-triple ([MU 2]) (η, h, η') with $h \in \mathfrak{g}_0, \eta' \in \mathfrak{g}_{-1}$ then $H_\eta = N_\eta.(H_\eta)_h$ with $N = exp(ad(\oplus_{i\geq 1} E_i(h) \cap \mathfrak{g}_0))$.

To prove the absolute convergence of T_η , it is enough to prove the absolute convergence of

$$\int_{P_{\mathbb{A}}/(H_{\mathbb{A}})_\eta} f(y.\eta)\Delta_{P_{\mathbb{A}}}^{-1}(y)dy$$

where $P = N.H_h$ is a parabolic subgroup of H and with this integral we arrive at $\int_{P_{\mathbb{A}}.\eta} f(z)|P_h(z)|^\beta dz$,with β to compute , so we have to look to the adelic Zeta function associated to a "commutative" PV of PT , and for $d \geq 2$ it is easy to prove the absolute convergence because we have an Ono-integral with $\beta > 0$ ([ONO]). □

Remark : in the case $d = 2$ $(H_\eta)_h$ has a nontrivial center of dimension one if $rank(\eta) = 2$ (this also true for the "commutative" case with $d = 2$)
For $j = 1, 2, 3$ let

$$T_j = \int_{H_{\mathbb{A}}/H} \Big(\sum_{\{\eta\in\mathfrak{g}_1 \ ,rk(\eta)=j\}} f(g\eta) \Big)dg_{\mathbb{A}} = \sum_{\{\eta\in\mathfrak{g}_1 \ ,rk(\eta)=j\}} \tau(H_\eta)T_\eta$$

then $\forall g \in G_{\mathbb{A}}$ $T_j(f(g.)) = |\chi(g)|^{-j\frac{d}{2}-\frac{1}{2}[\frac{j+1}{2}]}T_j(f)$.
We have the same results on $\mathfrak{g}_{-1}$. So we define in the same manner T_j^* and $\forall g \in G_{\mathbb{A}}$ $T_j^*(f(g.)) = |\chi(g)|^{j\frac{d}{2}+\frac{1}{2}[\frac{j+1}{2}]}T_j(f)$ (recall that in the dual PV the associated character is χ^{-1}).

With the proposition 4 it is not difficult to establish the following result originally due to Mars in the "commutative case "of PV of PT corresponding to $d = 8$ ([MA])

COROLLARY 5 (MARS). — *We assume that $\Delta = F_4$ or if $\Delta \neq F_4$ then $d \in \{4, 8\}$. For g' in $G_{\mathbb{A}}$ and f in $S(\mathfrak{g}_1)$, let*

$$U_f(g') = \int_{H_{\mathbb{A}}/H} \Big(\sum_{\xi\in\mathfrak{g}_1'} f(g'g\xi) \Big)dg_{\mathbb{A}}$$

then $\forall N > 0$ $\exists C_N$ such that $|U_f(g')| \leq C_N.|\chi(g')|^{-N}$ for large $|\chi(g')|$.

All the precedent results are true on the dual PV $(G, \mathfrak{g}_{-1})$ after the change of χ by χ^{-1}.

COROLLARY 6. — *We assume that $\Delta = F_4$ or if $\Delta \neq F_4$ then $d \in \{4, 8\}$ For every non singular η_0 we have $\tau'(H_{\eta_0}) = \frac{1}{2c_{\mathbb{F}}}\tau(H)$.*

We introduce first the usual notation : let $F^+(t) = 0$ if $|\, t\,|_{\mathbb{A}} < 1$ and else 1, $F^-(t) = 1 - F^+(t)$

$$W_f^+(\lambda, s) = W_{f.F^+\circ P}(\lambda, s) \qquad W_f^-(\lambda, s) = W_{f.F^-\circ P}(\lambda, s)$$

Proof

It is the method of [MA]. Let Ω the set of G-orbits in the non singular elements of $\mathfrak{g}_1 : \mathfrak{g}_1'$, f a Schwartz function such that $f(x) = 0$ for $x \in \cup_{\eta \in \Omega - \eta_0} G_{\mathbb{A}}.\eta$ and $\int_{\mathfrak{g}_{1,\mathbb{A}}} f(x)dx \neq 0$ (lemme 10 of [MA]) then

$$\sum_{\xi \in \mathbb{F}^*} M_{f(g_0.)}(\xi) = \frac{1}{\tau'(H_{\eta_0})} \int_{H_{\mathbb{A}}/H} (\sum_{x \in \mathfrak{g}_1'} f(g_0 g x)\,) dg_{\mathbb{A}}$$

We apply the ordinary Poisson formula :

$$\sum_{x \in \mathfrak{g}_1'} f(g_0 g x) = |\chi(g_0)|^{-\kappa}\ (\sum_{y \in \mathfrak{g}_{-1}} \hat{f}_{g_0 g}(y)\,) - (\sum_{x \in \mathfrak{g}_1 - \mathfrak{g}_1'} f_{g_0}(gx)\,)$$

With the corollary 5, $W_f^+(\lambda, .)$ is analytic on $\mathbb{C}$ so with proposition 1 $lim_{s \to 0, s \in S_A} sW_f^-(Id, s) = c_{\mathbb{F}}\rho_{\mathbb{F}}\hat{f}(0)$ (S_A as in proposition 1) but for $s \in S_A$ we have

$$s.\tau'(H_{\eta_0}).W_f^-(Id, s) = s. \int_{\mathbb{A}^*/\mathbb{F}^*} f^-(|t|)|t|^{2(\kappa+s)}.$$

$$\Big[(\int_{H_{\mathbb{A}}/H} [\, |t|^{-2\kappa} (\sum_{y \in \mathfrak{g}_{-1}} \hat{f}_{g_0 g}(y)) - (\sum_{x \in \mathfrak{g}_1 - \mathfrak{g}_1'} f_{g_0}(gx)\,] dg_{\mathbb{A}} \Big] dt^*_{\mathbb{A}} \text{ with } |\chi(g_0)| = |t|^2$$

$$= s. \int_{\mathbb{A}^*/^*_{\mathbb{F}}} f^-(|t|)|t|^{2s} \Big[\int_{H_{\mathbb{A}}/H} (\sum_{y \in \mathfrak{g}'_{-1}} \hat{f}(g_0 g y)\,) dg_{\mathbb{A}} \Big] dt^*_{\mathbb{A}} + \sum_{j=1}^{3} \frac{sT_j^*(\hat{f})\rho_{\mathbb{F}}}{2s + jd + [\frac{j+1}{2}]}$$

$$+ \frac{\rho_{\mathbb{F}}\tau(H)\hat{f}(0)}{2} - s[\sum_{j=1}^{3} \frac{T_j(f)\rho_{\mathbb{F}}}{2s + 2\kappa - jd - [\frac{j+1}{2}]} + \frac{\rho_{\mathbb{F}} f(0)}{2s + 2\kappa}\,]$$

and with the corollary 5 applied to the PV $(G, \mathfrak{g}_{-1})$ we obtain when we take the limit of the two members for $s \in S_A \to 0$: $\tau(H) = 2\tau'(H_{\eta_0})c_{\mathbb{F}}$. □
Finally we have obtained : (***)

$$\sum_{\xi \in \mathbb{F}^*} M_f(t^2\xi) = \frac{2c_{\mathbb{F}}}{\tau(H)}|t|^{2(\kappa-1)} \cdot \int_{H_{\mathbb{A}}/H} \Big(\sum_{x \in \mathfrak{g}_1'} f(g'gx) \Big) dg_{\mathbb{A}} \quad \text{with} \quad \chi(g') = t^2$$

3 The result

THEOREM 6. — *We assume that $\Delta = F_4$ or if $\Delta \neq F_4$ then $d \in \{4, 8\}$*
Let f a Schwartz function on $\mathfrak{g}_1$ and λ a character of $\mathbb{A}^*$, trivuial on $\mathbb{F}^*$, almost everywhere non ramified then
1) $W_f^+(\lambda, .)$ is analytic on $\mathbb{C}$
2) $W_f^-(\lambda, .)$ has an meromorphic extension given by the following formula

$$W_f^-(\lambda, s) = W_{\hat{f}}^+(\lambda^{-1}, -s-2-\frac{3}{2}d) - \delta_\lambda \frac{f(0)}{s+2+\frac{3}{2}d} + \delta_\lambda \frac{\hat{f}(0)}{s}$$

$$-\frac{\delta_\lambda}{\tau(H)}\Big(\frac{T_1(f)}{s+d+\frac{3}{2}} + \frac{T_2(f)}{s+\frac{d}{2}+\frac{3}{2}} + \frac{T_3(f)}{s+1}\Big)$$

$$+\frac{\delta_\lambda}{\tau(H)}\Big(\frac{T_1^*(\hat{f})}{s+\frac{d}{2}+\frac{1}{2}} + \frac{T_2^*(\hat{f})}{s+d+\frac{1}{2}} + \frac{T_3^*(\hat{f})}{s+\frac{3d}{2}+1}\Big)$$

with $\delta_\lambda = c_{\mathbb{F}}\rho_{\mathbb{F}}$ if $\lambda/(\mathbb{A}^*)^1 = id$ and 0 else
3) Functional equation : $W_f(\lambda, s) = W_{\hat{f}}(\lambda^{-1}, -s - \frac{3}{2}d - 2)$

Proof
1) is due to formula (***) and corollary 5.
2) is the same as the proof of corollary 6 because of the formula (***)
3) Comes from 2. □

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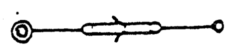
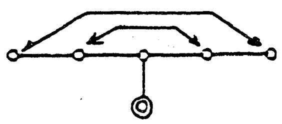
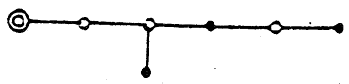
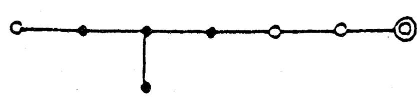
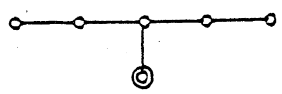
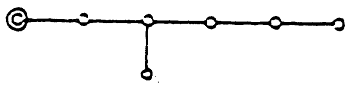
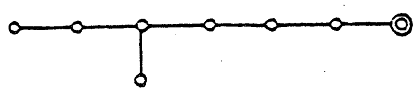
Table 1

List of PV of PT having root system $(R, \alpha_0) = (F_4, \alpha_1)$ given by their Satake diagram with the different values of d (cf. section 1 III) and the corresponding number in the algebraically closed case in M.Sato and T.Kimura- classification ([S-K]). The theorem 6 is verified for all of them except one split case with Dynkin Diagram $(\Delta, \lambda_0) = (E_6, \alpha_2)$.

We recall that ([S-K])
(5) : $(GL(6), \Lambda_3, V(20))$
(14) : $(GL(1) \times Sp(3), \square \times \Lambda_3, V(1) \times V(14))$
(23) $(GL(1) \times Spin(12), \square \times half\ spin\ rep., V(1) \times V(32))$
(29) $(GL(1) \times E_7, \square\times, V(1) \times \Lambda_6, V(1) \times V(56))$ (case of [IG 2])

(Δ, λ_0)	Satake diagram	d	number in [S-K] class.	In th.6
(F_4, α_1)		1	(14)	X
		2	(5)	X
		4	(23)	X
		8	(29)	X
(E_6, α_2)		2	(5)	
(E_7, α_1)		4	(23)	X
(E_8, α_8)		8	(29)	X